13/12/17

Τύποι Αριθμ. Ολοκλήρωσης των Newton-Cotes.

Θεωρούμε την ομοιόμορφη διαμέριση του $[a,b]$.
$x_0, x_1, \dots, x_n$, όπου $x_i = a + ih$, $h = \frac{b-a}{n}$

Προσεγγίζουμε το $I(f) = \int_a^b f(x)\,dx$ με $Q_{n+1}(f) = \int_a^b P_n(x)\,dx$,

P_n το πολυώνυμο παρεμβολής της f στα $x_0, x_1, \dots, x_n$
$Q_n(f) = w_0 f(x_0) + w_1 f(x_1) + \dots + w_n f(x_n)$, όπου

$$w_i = \int_a^b L_i(x)\,dx \quad , \quad L_i(x) = \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{x-x_j}{x_i-x_j} \text{ , πολυώνυμο Lagrange.}$$

$$w_i = \int_a^b \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{x-x_j}{x_i-x_j}\,dx = \int_a^b \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{a+s\cdot h-(a+jh)}{a+ih-(a+jh)}\,d(a+sh) \quad , 0\le s\le 1$$

$$= h\int_0^n \prod_{\substack{j=0\\ j\neq 0}}^{n} \frac{s-j}{i-j}\,ds$$

Οι συντελεστές w_i εμφανίζονται συμμετρικά στον τύπο δηλαδή $w_{n-i} = w_i$.

$$w_{n-i} = h\int_0^n \prod_{\substack{j=0\\ j\neq n-i}}^{n} \frac{s-j}{n-i-j}\,ds \text{ , θεωρούμε } s = n-t \text{ τότε}$$

$$w_{n-i} = h\int_0^n \prod_{\substack{j=0\\ j\neq n-i}}^{n} \frac{n-t-j}{n-i-j}\,d(n-t) = -h\int_n^0 \prod_{\substack{j=0\\ j\neq n-i}}^{n} \frac{t-(n-j)}{i-(n-j)}\,dt =$$

$l=(n-j)$

$$= h\int_0^n \prod_{\substack{l=0\\ l\neq i}}^{n} \frac{t-l}{i-l}\,dt = w_i$$

$j \neq n-i \Leftrightarrow i \neq n-j = l$.

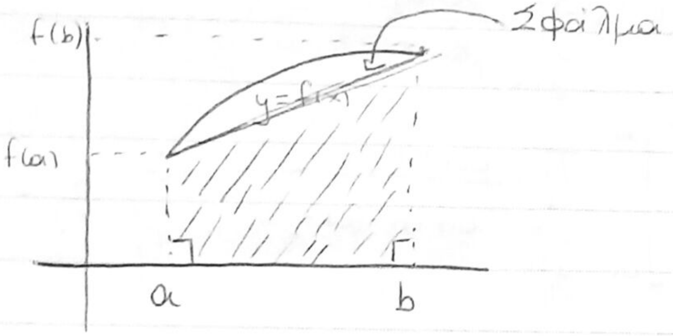
∘ $n=1$. Τύπος ή Κανόνας του Τραπεζίου

Θεωρούμε $x_0=a$, $x_1=b=a+h$, $h=b-a$.

$$Q_2(f) = w_0 f(x_0) + w_1 f(x_1)$$

$$w_0 = h\int_0^1 \frac{s-1}{0-1}\,ds = h\int_0^1 (1-s)\,ds = h\left[s-\frac{s^2}{2}\right]_0^1 = h\left(1-\frac{1}{2}\right) = \frac{h}{2}$$

$$Q_2(f) = \frac{h}{2}f(x_0) + \frac{h}{2}f(x_1) = \frac{h}{2}\left(f(x_0)+f(x_1)\right)$$



∘ $n=2$. Τύπος ή Κανόνας του Simpson.

Θεωρούμε $x_0=a$, $x_1=a+h$, $x_2=a+2h$, $h=\frac{b-a}{2}$

$$Q_3(f) = w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2)$$

$$\boxed{w_i = h\int_0^n \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{s-j}{i-j}\,ds}$$

$$w_0 = h\cdot\int_0^2 \frac{(s-1)(s-2)}{(0-1)(0-2)}\,ds = \frac{h}{2}\int_0^2 (s^2-3s+2)\,ds =$$

$$= \frac{h}{2}\left[\frac{s^3}{3}-\frac{3s^2}{2}+2s\right]_0^2 = \frac{h}{2}\left(\frac{8}{3}-\frac{12}{2}+4\right) = \frac{h}{2}\left(\frac{8}{3}-2\right)$$

$$= \frac{h}{2}\left(\frac{2}{3}\right) = \frac{h}{3}$$

$$W_1 = h\int_0^2 \frac{(s-0)(s-2)}{(1-0)(1-2)}\,ds = -h\int_0^2 (s^2-2s)\,ds = -h\left[\frac{s^3}{3}-\frac{2s^2}{2}\right]_0^2 =$$

$$-h\left(\frac{8}{3}-4\right) = -h\left(-\frac{4}{3}\right) = \frac{4h}{3}$$

$$W_2 = W_0 = \frac{h}{3}$$

$$Q_3(f) = \frac{h}{3}f(x_0) + \frac{4}{3}hf(x_1) + \frac{h}{3}f(x_2) = \frac{h}{3}\left(f(x_0)+4f(x_1)+f(x_3)\right)$$

- $n=3$ Τύπος ή κανόνας των $3/8$

Θεωρούμε $x_0 = a$, $x_1 = a+h$, $x_2 = a+2h$, $x_3 = a+3h = b$, $h = \frac{b-a}{3}$

$$Q_4(f) = w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$$

$$W_0 = h\int_0^3 \frac{(s-1)(s-2)(s-3)}{(0-1)(0-2)(0-3)}\,ds = -\frac{h}{6}\int_0^3 (s^3-6s^2+11s-6)\,ds =$$

$$= \frac{h}{6}\left[\frac{s^4}{4} - 6\cdot\frac{s^3}{3} + 11\frac{s^2}{2} - 6s\right]_0^3 = -h/6\left(\frac{81}{4} - 54 + \frac{99}{2} - 18\right)$$

$$= -\frac{h}{6}\,\frac{279-288}{4} = \frac{3h}{8}$$

$$W_1 = h\int_0^3 \frac{s(s-2)(s-3)}{(1-0)(1-2)(2-3)}\,ds = \frac{h}{2}\int_0^3 (s^3-5s^2+6s)\,ds$$

$$= \frac{h}{2}\left[\frac{s^4}{4} - 5\frac{s^3}{3} + 6\frac{s^2}{2}\right]_0^3 = \frac{h}{2}\left(\frac{81}{4} - 45 + 27\right) = \frac{h}{2}\,\frac{81-72}{4} = \frac{9}{8}h.$$

$$W_2 = W_1 = \frac{9}{8}h, \quad W_3 = W_0 = \frac{3}{8}h$$

$$Q_4(f) = \frac{3}{8}h\,f(x_0) + \frac{9}{8}h\,f(x_1) + \frac{9}{8}h\,f(x_2) + \frac{3}{8}h\,f(x_3)$$

$$= \frac{3}{8}h\left(f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right)$$

$$I(x^3) = \int_a^b x^3 dx = \left[\frac{x^4}{4}\right]_a^b = \frac{b^4-a^4}{4}$$

$$Q_3(x^3) = \frac{h}{3}\left(f(a) + 4(a+h)^3 + (a+2h)^3\right) \Rightarrow$$

$$Q_3(x^3) = \frac{h}{3}\left(a^3 + 4(a+h)^3 + (a+2h)^3\right) =$$

$$= a^3 + 4(a^3+3a^2h+3ah^2+h^3) + a^3 + 6a^2h + 12ah^2 + 8h^3) =$$

$$= \frac{h}{3}(6a^3 + 18a^2h + 24ah^2 + 12h^2) =$$

$$= h(2a^3 + 6a^2h + 8ah^2 + 4h^3)$$

$$b^4 - a^4 = 2h(b^3 + b^2a + ba^2 + a^3) = 2h((a+2h)^3 + (a+2h)^2a + (a+2h)a^2 + a^3)$$

$$= 2h(a^3 + 6a^2h + 12ah^2 + 8h^3 + a^3 + 4a^2h + 4ah^2 + a^3]$$

$$= 2h(4a^3 + 12a^2h + 16ah^2 + 8h^3)$$

$$\frac{b^4-a^4}{4} = 2h(a^3 + 3a^2h + 4ah^2 + 2h^3) = h(2a^3 + 6a^2h + 8ah^2 + 4h^3)$$